University of California, Berkeley
Physics 110B Spring 2001   Section 1 (*Strovink*)

SOLUTION TO FINAL EXAMINATION

Directions. Do all six problems, which have unequal weight. This is a closed-book closed-note exam except for three $8\frac{1}{2} \times 11$ inch sheets containing any information you wish on both sides. Calculators are not needed. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. *You must justify what you do or say.* Express your answer in terms of the quantities specified in the problem. Box or circle your answer. Remember that when you are asked for the value of a vector quantity, you must supply both the magnitude and direction.

1. (40 points)
The total power $P(t)$ radiated by an ideal electric dipole $\mathbf{p}(t)$ is given by the Larmor formula

$$P(t) = \frac{1}{4\pi\epsilon_0}\frac{2|\ddot{\mathbf{p}}(t_{\rm ret})|^2}{3c^3} ,$$

where $t_{\rm ret}$ is the retarded time.

(a) (15 points) Consider a single positive charge e located at position $(x, y, z) = (d, 0, d\cos\omega t)$, where d and ω are constants. Approximate $d \ll \lambda$, where λ is the vacuum wavelength of the emitted radiation. Working to second order in the small quantity d/λ, compute the *time-averaged* power $\langle P\rangle$ radiated by this charge.
Solution:
Applying the Larmor formula to an electric dipole

$$\begin{aligned}
P &= \frac{1}{4\pi\epsilon_0}\frac{2|\ddot{\mathbf{p}}(t_{\rm ret})|^2}{3c^3} \\
&= \frac{1}{4\pi\epsilon_0}\frac{2e^2d^2\omega^4\cos^2\omega t_{\rm ret}}{3c^3} \\
\langle P\rangle &= \frac{1}{4\pi\epsilon_0}\frac{e^2d^2\omega^4}{3c^3} .
\end{aligned}$$

(b) (10 points) How much time-averaged mechanical work per unit time $\langle dW/dt\rangle$ must be exerted upon this charge in order to keep it moving as specified in **(a)**?
Solution:
The mechanical work done on the charge per unit time would need to supply the power that it radiates. Thus

$$\langle dW/dt\rangle = \langle P\rangle = \frac{1}{4\pi\epsilon_0}\frac{e^2d^2\omega^4}{3c^3} .$$

This answer may also be obtained by considering the radiation reaction force on the charge.

(c) (15 points) A second *positive* charge e is added, located at position $(-d, 0, -d\cos\omega t)$. What is the new *time-averaged* power $\langle P'\rangle$ radiated by both charges? Continue to work only to second order in the small quantity d/λ.
Solution:
The second charge is located on the opposite side of the origin with respect to the first charge. Thus it cancels the electric dipole moment due to the first charge. Higher-order multipole radiation may remain, but such contributions will be raised to higher powers of d/λ. Therefore, to the same order in d/λ, $\langle P'\rangle$ vanishes.

2. (35 points)
A plane electromagnetic wave is described by

$$\mathbf{E}(z, t) = \mathrm{Re}\Big(\tilde{\mathbf{E}}\,\exp\big(i(kz - \omega t)\big)\Big) ,$$

where

$$\tilde{\mathbf{E}} = E_0\big((2 - i)\hat{\mathbf{x}} + (1 - 2i)\hat{\mathbf{y}}\big) ,$$

and E_0, k, and ω are real constants. A linear polarizer is placed in the beam, and oriented so that the largest possible fraction of the original beam's irradiance is transmitted. What is that fraction?

Solution
The beam is described by the (unnormalized) Jones vector

$$J \equiv \begin{pmatrix} 2-i \\ 1-2i \end{pmatrix} .$$

A linear polarizer with transmission axis oriented along the $\hat{\mathbf{x}}$ direction has the Jones matrix

$$M(0) \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} .$$

If the polarizer's transmission axis is oriented at angle ϕ with respect to the $\hat{\mathbf{x}}$ direction, it is represented by the Jones matrix

$$\begin{aligned} M(\phi) &= R^{-1}M(0)R \\ &= \begin{pmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{pmatrix} , \end{aligned}$$

where the two-dimensional rotation matrix is

$$R \equiv \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} .$$

Before the polarizer, the beam irradiance I is proportional to

$$\begin{aligned} I &\propto J^\dagger J \\ &= \begin{pmatrix} 2+i & 1+2i \end{pmatrix} \begin{pmatrix} 2-i \\ 1-2i \end{pmatrix} \\ &= (4+1)+(1+4) = 10 . \end{aligned}$$

After the polarizer, the irradiance I' is proportional to

$$\begin{aligned} I' &\propto (MJ)^\dagger MJ \\ &= J^\dagger(M^\dagger M)J . \end{aligned}$$

But $M^\dagger M = M$, as can easily be verified: $M^\dagger = M$, and adding a second ideal polarizer does nothing beyond the effect of the first, so $M^2 = M$. Thus

$$\begin{aligned} I &\propto J^\dagger MJ \\ &= \begin{pmatrix} 2+i & 1+2i \end{pmatrix} \times \\ &\times \begin{pmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{pmatrix} \begin{pmatrix} 2-i \\ 1-2i \end{pmatrix} \\ &= 5 + 8\sin\phi\cos\phi \\ &= 9 \text{ (max)} \end{aligned}$$

when $\phi = \pi/4$. Therefore, at maximum, $I'/I = 9/10$.

3. (35 points)
A plane wave $U_0 \cos(kz - \omega t)$ is incident normally on a screen. Fraunhofer conditions apply. The diffracted wave is observed from $z \to \infty$ at various angles θ with respect to the z axis.

(a) (15 points) Assume that the screen has three long parallel slits with equal spacing b and equal negligible width. Compute the irradiance ratio $I(\theta)/I(\theta = 0)$.
Solution:
In analogy to the standard double slit problem,

$$U(\theta) \propto 1 + e^{i\beta} + e^{-i\beta} ,$$

where $\beta = kb\sin\theta$. Therefore

$$\begin{aligned} &U(\theta) \propto 1 + 2\cos\beta \\ &\frac{I(\theta)}{I(0)} = \frac{(1+2\cos\beta)^2}{9} . \end{aligned}$$

This result is equivalent to $\frac{1}{9}\sin^2(3\gamma)/\sin^2\gamma$, where $\gamma = \beta/2$.

(b) (20 points) Instead assume that the screen has five long parallel slits with equal spacing b. The slit widths are still negligible; however, they are a function of the slit location, so that the five slit areas vary according to the ratio 1:2:3:2:1. Compute the irradiance ratio $I(\theta)/I(\theta = 0)$.
Solution:
This configuration is equivalent to a triple-superposition of the triple-slit problem in (**a**), with the characteristic spacing of the superposition equal to the characteristic spacing of the slit. Therefore it is a convolution of the arrangement in (**a**) with itself. Under Fraunhofer conditions, the image is a Fourier transform of the aperture function, and the Fourier transform of a convolution is the product of the individual Fourier transforms. Therefore

$$\frac{I(\theta)}{I(0)} = \frac{(1+2\cos\beta)^4}{81} .$$

This answer may also be obtained by the brute force methods of (**a**).

4. (20 points)
A Survivor contestant tries to signal a blimp hovering nearly overhead. It is pitch dark, and his only source of light is an infinitesimal, monochromatic, isotropic-light-emitting diode (LED). The naked LED isn't quite bright enough to be seen by his blimp-borne rescuer. Remembering Physics 110B, the contestant resolves to amplify the light signal that the rescuer perceives.

(a) (10 points) The contestant stretches a large opaque plastic sheet over a flat frame and pokes a small (couple of mm dia) circular hole in it. He carefully positions the hole directly between the LED and the blimp, separated from the LED by a couple of meters. Relative to the naked LED, is it possible that the irradiance seen by the rescuer increases? If so, by what maximum factor?
Solution:
The hole could consist of an odd number of Fresnel zones (one zone would be convenient, given the rough dimensions), in which case the irradiance seen by the rescuer would be boosted by a factor of ≈ 4.

(b) (10 points) Lacking a plastic sheet, the contestant disassembles his bicycle hub to obtain a small (couple of mm dia) blackened steel ball. Using a spiderweb thread, he carefully hangs the ball directly between the LED and the blimp, separated from the LED by a couple of meters. Relative to the naked LED, is it possible that the irradiance seen by the rescuer increases? If so, by what maximum factor?
Solution:
Using the edge of the ball (as opposed to $R = 0$) as the beginning of the first Fresnel zone, and adding up the contributions of the zones, the irradiance seen by the rescuer would be approximately the same as if the ball were removed. Therefore the irradiance seen by the rescuer would not increase.
This result can also be obtained by use of Babinet's argument.

5. (35 points)
In the Drüde model for electromagnetic wave propagation in a dilute material medium, electrons (of mass m and charge $-e$) satisfy the equation of motion

$$m\ddot{x} = -\gamma m\dot{x} - kx - eE_x \ ,$$

where γ is an effective damping constant, k is an effective spring constant, and E_x is an electric field component.

Working at a particular angular frequency ω, and defining the complex electric field $\tilde{E}_x$ and complex current density $\tilde{J}_x$ through

$$\begin{aligned} E_x &\equiv \mathrm{Re}\big(\tilde{E}_x \exp{(-i\omega t)}\big) \\ J_x &\equiv \mathrm{Re}\big(\tilde{J}_x \exp{(-i\omega t)}\big) \ , \end{aligned}$$

one can then define the complex conductivity $\tilde{\sigma}$ through

$$\tilde{J}_x \equiv \tilde{\sigma}\tilde{E}_x \ .$$

In a medium having N electrons/m^3 that are so weakly bound that k is negligible, use the above information to derive the complex conductivity $\tilde{\sigma}$ as a function of angular frequency ω.
[*Hint*: Define $x \equiv \mathrm{Re}\big(\tilde{x}\exp{(-i\omega t)}\big)$.]
Solution:
Substituting

$$x \equiv \mathrm{Re}\big(\tilde{x}\exp{(-i\omega t)}\big) \ ,$$

in the equation of motion, and neglecting k with respect to $\gamma m\omega$ in view of the negligibly weak binding, we obtain

$$\begin{aligned} -m\omega^2\tilde{x} &= i\gamma m\omega\tilde{x} - e\tilde{E}_x \\ \tilde{x} &= \frac{e\tilde{E}_x/m}{\omega^2 + i\gamma\omega} \ . \end{aligned}$$

Solving for $\tilde{J}_x$,

$$\begin{aligned} J_x &= -eN\dot{x} \\ \Rightarrow \tilde{J}_x &= i\omega eN\tilde{x} \\ &= \frac{i\omega Ne^2\tilde{E}_x/m}{\omega^2 + i\gamma\omega} \\ \tilde{\sigma} &\equiv \frac{\tilde{J}_x}{\tilde{E}_x} \\ &= \frac{Ne^2/m}{\gamma - i\omega} \ . \end{aligned}$$

6. (35 points)
A point charge e travelling on the z axis has position

$$\mathbf{r}(t) = +\hat{\mathbf{z}}\beta ct \quad (t < 0)$$
$$= -\hat{\mathbf{z}}\beta ct \quad (t > 0) ,$$

where β is a positive constant that is not $\ll 1$. That is, the charge reverses direction instantaneously at $t = 0$, while it is at the origin. The fields that the charge produces are viewed by an observer at $(x, 0, 0)$, where $x > 0$.

(a) (20 points) What magnetic field $\mathbf{B}$ does the observer see at $t = 0$?
Solution:
At $t = 0$, the magnetic field observed at $(x, 0, 0)$ was produced by the charge when it was at $t_{\rm ret} < 0$, when it was still moving in the positive z direction. Therefore this is the field of a uniformly moving charge. To evaluate it, we first obtain the field in a (primed) coordinate system with its origin attached to the charge. In the primed system, the observer is located at the coordinates $(x', y', z') = (x, 0, \gamma z - \gamma\beta ct) = (x, 0, 0)$. There the (purely electrostatic) field is given by

$$\mathbf{E}' = \hat{\mathbf{x}}\frac{e}{4\pi\epsilon_0 x^2} .$$

In the lab frame, using the Lorentz transformation for electromagnetic fields, the magnetic field is given by

$$\begin{aligned} B_{\parallel} &= B'_{\parallel} = 0 \\ c\mathbf{B}_{\perp} &= \gamma c\mathbf{B}'_{\perp} + \gamma\vec{\beta} \times \mathbf{E}'_{\perp} \\ &= 0 + \gamma\beta\frac{e}{4\pi\epsilon_0 x^2}\hat{\mathbf{z}} \times \hat{\mathbf{x}} \\ \mathbf{B}_{\perp} &= \frac{\gamma\beta}{c}\frac{e}{4\pi\epsilon_0 x^2}\hat{\mathbf{y}} . \end{aligned}$$

At this observation point the $\hat{\mathbf{y}}$ direction is the same as the ϕ direction, as one expects.

This answer may also be obtained by using the standard expressions for the electromagnetic field of a uniformly moving point charge, *e.g.* Griffiths 10.68-10.69.

(b) (15 points) At time t such that $ct = x$ (exactly!), what is the direction of the electric field $\mathbf{E}$ seen by the observer? (You need consider only the part of the total electric field which is dominant at exactly that time.) Justify your answer.
Solution:
At $t = x/c$, the retarded time is $t_{\rm ret} = 0$. So the fields seen by the observer are the fields of a charge that is reversing the direction of its velocity (with infinite acceleration in this case). Therefore the fields at this time are dominated by the acceleration fields. For a charge accelerating along $\hat{\mathbf{z}}$, $\mathbf{E}$ is in the $\hat{\theta}$ direction, or $-\hat{\mathbf{z}}$ for this observer. However, in this problem the charge accelerates in the $-\hat{\mathbf{z}}$ direction, so $\mathbf{E}$ is along $+\hat{\mathbf{z}}$.